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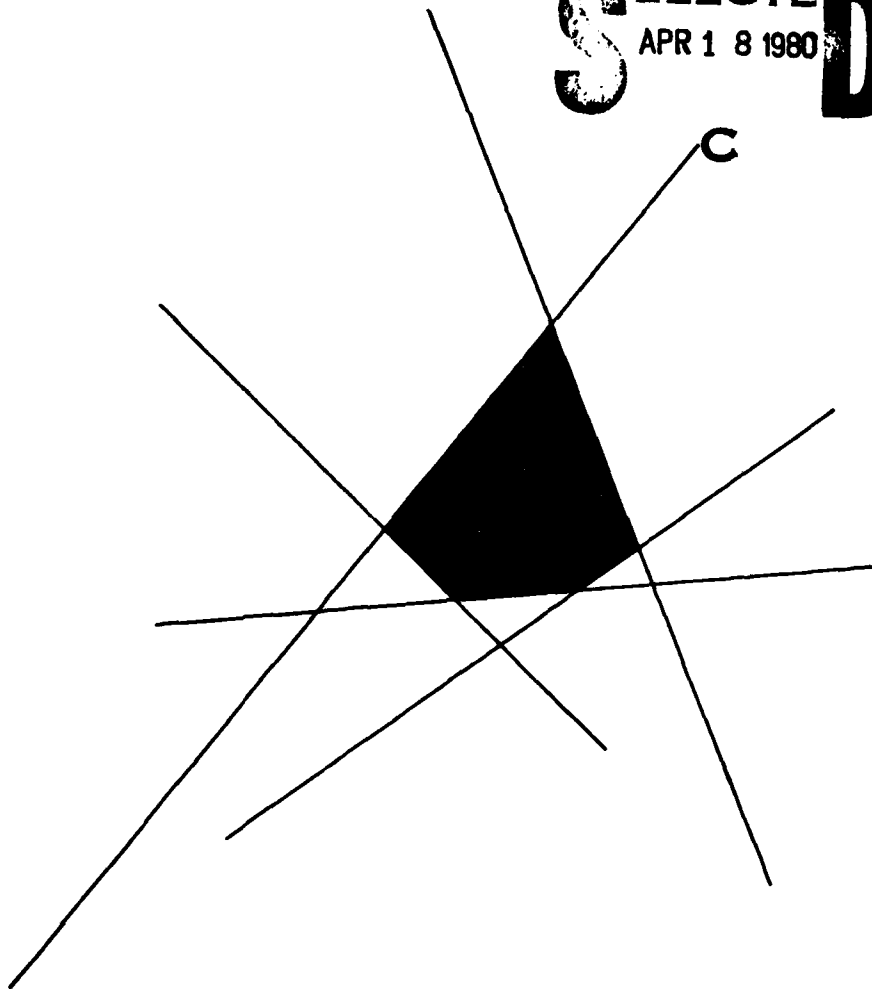
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**BAYESIAN NONPARAMETRIC STATISTICAL INFERENCE  
FOR SHOCK MODELS AND WEAR PROCESSES**

by  
ALBERT Y. LO

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ADDENDUM  
TO THE  
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INSTRUCTIONS  
FOR THE  
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# ABSTRACT

Statistical procedures for shock models and wear processes are considered in this paper. We show that independent gamma-Dirichlet priors are conjugate priors when sampling from these shock models. Bayes rules given the observations are computed. In particular, we calculate the Bayes estimates of the survival probabilities for these models. We show consistency of the posterior distribution as well as weak convergence of the centered and suitably rescaled posterior processes.

# BAYESIAN NONPARAMETRIC STATISTICAL INFERENCE FOR SHOCK MODELS AND WEAR PROCESSES

by

Albert Y. Lo

## 1. INTRODUCTION AND SUMMARY

Let a device be subject to shocks occurring randomly in time according to a homogenous Poisson point process  $N = \{N(t) ; t \in R\}$  with intensity  $\lambda$ . The  $i^{\text{th}}$  shock causes a random amount of damage  $X_i$ ,  $i = 1, \dots, n, \dots$ . The random damages are assumed to be independent and identically distributed as  $F$ , where  $F$  is a distribution function supported by  $[0, \infty)$ . (This implies damages are never negative, also note that the results in Section 2 do not depend on the support of  $F$ .) This shock model have been studied by Esary, Marshall and Proschan (1973), Barlow and Proschan (1975), among others. The analogy of the shock model in risk and acturial analysis has been given by Bühlmann (1970, Chapter 2).

We consider Bayesian nonparametric procedures for the shock model and assume that the reader is familiar with the results in Ferguson (1973). The prior distribution we use for  $(\lambda, F)$  is the "independent gamma-Dirichlet" prior. By this we mean that  $\lambda$  has a gamma distribution with parameter  $\gamma$  and  $\theta$ , denoted by  $U_{\gamma, \theta}(d\lambda)$ ,  $F$  has a Dirichlet distribution with parameter  $\alpha$ , denoted by  $P_{\alpha}(dF)$ , and  $\lambda$  and  $F$  are independent. We write  $(\lambda, F) \sim U_{\gamma, \theta} \times P_{\alpha}$  and find that the posterior distribution of  $(\lambda, F)$  given the Poisson process up to time  $T$  and  $X_1, \dots, X_{N(T)}$  is again an independent gamma-Dirichlet distribution, namely,  $U_{\gamma+N(T), \theta+T} \times P_{\alpha + \sum_{x_1} \delta_{x_1}}$ , where the summation is from one to  $N(T)$  and  $\delta_x$  denotes the probability measure degenerate at  $x$ .



The asymptotic properties of the posterior distribution is considered and it is shown that the posterior limiting law of the parameters centered and suitably rescaled is Gaussian.

A parameter of primary interest is the survival probability, which is the probability that the device survives beyond time  $t$ . We calculate the Bayes estimate for this parameter.

## 2. THE PRIOR AND POSTERIOR DISTRIBUTIONS

Throughout this paper, we assume that the damages  $X_1, \dots, X_k, \dots$  caused by the shocks are independent and identically distributed random variables having distribution  $F$  and that the shock process

$N_T = \{N(t) ; t \in [0, T]\}$  is a Poisson process with parameter  $\lambda$  which is independent of the damages. The observables are therefore  $N_T$  and  $X_T = (X_1, \dots, X_{N(T)})$ .

We formalize the above assumptions as follows: for each measurable partition  $A_1, \dots, A_l$  of  $[0, T]$  and each collection  $C_1, C_2, \dots$  of Borel sets on the real line, the joint distribution of  $N_T$  and  $X_T$  for given  $(\lambda, F)$  is determined by

$$\begin{aligned} P_{\lambda, F} \{N_T(A_j) = k_j ; j = 1, \dots, l, X_i \in C_i ; i = 1, \dots, N(T)\} \\ (2.1) \quad = e^{-\lambda T N(T)} \cdot \prod_{j=1}^l \frac{m^{k_j}(A_j)}{k_j!} \cdot \prod_{i=1}^{N(T)} F(C_i) \end{aligned}$$

where  $N_T(A_j)$  is the number of shocks in  $A_j$ ,  $m$  is the Lebesgue measure and  $k_1, \dots, k_l$  are nonnegative integers with

$$\sum_{j=1}^l k_j = N(T) .$$

Using the idea of conjugate priors, we let  $\lambda \sim U_{\gamma, \theta}$  where  $U_{\gamma, \theta}$  is the gamma distribution with mean  $\gamma/\theta$  and variance  $\gamma/\theta^2$  and let  $F \sim P_\alpha$  where  $P_\alpha$  is the Dirichlet distribution of  $F$  with parameter  $\alpha$  where  $\alpha$  is the finite measure on the reals. With the additional assumption that  $\lambda$  and  $F$  are independent, we have the independent gamma-Dirichlet distribution  $U_{\gamma, \theta} \times P_\alpha$  of  $(\lambda, F)$ .

We need the following notations to prove the main theorem. Let  $Q_\lambda$  be the distribution of the observations given  $\lambda$ ,  $Q_F$  the distribution of the observations given  $F$  and  $Q$  the marginal distribution of the observations. The characterizations of these distributions via finite dimensional sets are given in the appendix. We are ready to present the following

Theorem 2.1:

Given  $N_T$  and  $X_T$ , the (posterior) distribution of

- (i)  $\lambda$  is  $U_{\gamma+N(T), \theta+T}$
- (ii)  $F$  is  $P_{N(T) + \sum_{i=1}^{\alpha} \delta_{x_i}}$  and
- (iii)  $\lambda$  and  $F$  are independent.

Proof:

Note that according to (2.1), given  $(\lambda, F)$ , the conditional distribution of  $N_T$  and  $X_T$  given  $N(T)$  and  $X_T$  is independent of  $(\lambda, F)$ . Thus  $\{N(T), X_T\}$  is a set of sufficient statistics for the family of model distributions  $\{P_{\lambda, F} : (\lambda, F) \in R^+ \times \mathcal{H}\}$ . Thus, we can treat the reduced problem and hence, (2.1) becomes

$$\begin{aligned}
 &P_{\lambda, F}\{N(T) = k, X_i \in C_i; i = 1, \dots, N(T)\} \\
 (2.1') \quad &= \frac{e^{-\lambda T} (\lambda T)^k}{k!} \prod_{i=1}^k F(C_i).
 \end{aligned}$$

To prove (i), we first hold  $F$  fixed (that is given  $F$ ). We shall show that the posterior distribution of  $\lambda$  given  $N(T)$  and  $X_T$  is

$u_{\gamma+N(T), \theta+T}^{(d\lambda)}$ , which is independent of  $F$ . We need the following notations.

Let  $A = \{N(T) = k\}$  and  $\underline{C} = \prod_{i=1}^k C_i$ . According to the definition of conditional distributions, we have to check that for all  $B \in \mathcal{B}$ , for all such  $A$ 's and for all such  $\underline{C}$ 's, the following equality holds

$$\int_A \int_{\underline{C}} u_{\gamma+N(T), \theta+T}^{(B)} Q_F(dN(T), d\underline{x}) = \int_B P_{\lambda, F}(A \times \underline{C}) u_{\gamma, \theta}^{(d\lambda)},$$

where  $Q_F$  is given by (A.1). Since  $Q_F(\underline{C}) = \prod_{i=1}^k F(C_i)$  is cancelled out in both sides given  $F$ , we need only check that

$$\int_A u_{\gamma+N(T), \theta+T}^{(B)} Q_F(dN(T)) = \int_B P_{\lambda, F}(A) u_{\gamma, \theta}^{(d\lambda)}.$$

But this is well known since it says that gamma distributions are conjugate priors when sampling from Poisson random variables. Thus (i) holds and given  $F$  the posterior distribution of  $\lambda$  is independent of  $F$ .

Note that (ii) can be proved similarly. We first hold  $\lambda$  fixed, and check that  $P_{N(T)}^{(\alpha + \sum_{i=1}^N \delta_{x_i})}(dF)$  satisfies the definition of the posterior

distribution of  $F$  given  $N_T$ ,  $X_T$  and  $\lambda$ . We need to check if the following equality holds for all  $D \in \mathcal{A}$ , all  $A$ 's and all  $\underline{C}$ 's, where  $Q_\lambda$  is given by (A.1)

$$\int_A \int_{\underline{C}} P_{N(T)}^{(\alpha + \sum_{i=1}^N \delta_{x_i})} (D) Q_\lambda(dN(T), d\underline{x}) = \int_D P_{\lambda, F}(A \times \underline{C}) P_\alpha(dF).$$

Since  $Q_\lambda(A) = \frac{e^{-\lambda T} (\lambda T)^k}{k!}$  is cancelled out in both sides, we need only to check if

$$\int_C P_{\alpha + \sum_{i=1}^k \delta_{x_i}} (D) Q_\lambda(dx) = \int_D P_{\lambda, F}(C) P_\alpha(dF) .$$

But then this is true by Ferguson (1973, Theorem 1 of Section 3).

We have also shown that given  $\lambda$ , the posterior distribution of  $F$  is independent of  $\lambda$ , thus (iii) is true. ||

### 3. LIMITING DISTRIBUTION

The consistency of the posterior distributions is well known. A general proof of this phenomenon is given by Doob (1949) who uses martingale methods to obtain the result that the posterior distribution converges to the distribution degenerate at the true parameter under which the sample is actually drawn, for almost all true parameters. The delicate problem of the classification of the true parameters under which consistency holds is treated by many authors in various cases. In particular, if the parameters are discrete distribution functions then the convergence holds for all true parameters lying in the support of the so-called tail free prior probabilities, see Freedman (1963) and Fabius (1964). Berk (1970) investigates this problem in the dominated case.

In this section, we would rather search for the limiting posterior distribution for the parameters. Note that in the parametric case, this problem has been considered by many authors. See LeCam (1963), or more recently, Walker (1969) and Dawid (1970). Our problem is non-parametric in nature and our results are based on a theorem of Lo (1978a).

We denote the "true" values of the intensity  $\lambda$  and the damage distribution by  $\lambda_0$  and  $F_0$  respectively. Thus,  $P_{\lambda_0, F_0}$  is the true probability distribution of the observations. We denote the "posterior random variable"  $\lambda$  and the "posterior Dirichlet process"  $F$  given observations up to time  $T$  by  $\lambda_T$  and  $F_T$  respectively. Thus, a.s.  $\left[ P_{\lambda_0, F_0} \right]$ ,

$$(\lambda_T, F_T) \sim u_{\gamma+N(T), \theta+T} \times P_{\alpha + \sum_{i=1}^{N(T)} \delta_{x_i}}.$$

The following proposition shows the consistency of the posterior distributions.

Proposition 3.1:

The posterior distribution of  $\lambda$  converges to the distribution degenerate at  $\lambda_0$  and the posterior distribution of  $F$  converges to the distribution degenerate at  $F_0$ , as  $T$  goes to infinity, a.s.  $\left[ P_{\lambda_0, F_0} \right]$ .

Proof:

Note that  $E(\lambda_T \mid N_T, X_T) = \frac{\gamma + N(T)}{\theta + T}$  and  $E\left\{\left(\lambda_T - \frac{\gamma + N(T)}{\theta + T}\right)^2 \mid N_T, X_T\right\} = \frac{\gamma + N(T)}{(\theta + T)^2}$ . Because  $\frac{N(T)}{T} \rightarrow \lambda_0$  a.s.  $\left[ P_{\lambda_0, F_0} \right]$  via some renewal arguments, see Chung (1974, p. 134). Thus the posterior mean of  $\lambda$  goes to  $\lambda_0$  and the posterior variance of  $\lambda$  goes to zero, thus the first assertion follows. To show the second one, let  $B_j$ ;  $j = 1, \dots, K$  be a measurable partition of  $R$ , we shall show that a.s.  $\left[ P_{\lambda_0, F_0} \right]$ .

$$(1) \quad E(F_T(B_j) \mid N_T, X_T) \rightarrow F_0(B_j); \quad j = 1, \dots, K.$$

$$(2) \quad E\left\{\left(F_T(B_j) - \frac{\left[\alpha + \sum_{i=1}^{N(T)} \delta_{x_i}\right](B_j)}{\alpha(R) + N(T)}\right)\left(F_T(B_\ell) - \frac{\left[\alpha + \sum_{i=1}^{N(T)} \delta_{x_i}\right](B_\ell)}{\alpha(R) + N(T)}\right) \mid N_T, X_T\right\} \rightarrow 0;$$

$j, \ell = 1, \dots, K.$

Then the result will follow via Kallenberg (1976, Theorem 4.3). Now,

(1) holds because as  $T$  goes to infinity,

$$E\{F_T(B_j) \mid N_T, X_T\} = \frac{\alpha(B_j) + \sum_{i=1}^{N(T)} \delta_{x_i}(B_j)}{\alpha(R) + N(T)} \rightarrow F_0(B_j) ; j = 1, \dots, K.$$

(2) holds because for  $j \neq \ell$

$$\begin{aligned} & E \left\{ \left( F_T(B_j) - \frac{\alpha(B_j) + \sum_{i=1}^{N(T)} \delta_{x_i}(B_j)}{\alpha(R) + N(T)} \right) \left( F_T(B_\ell) - \frac{\alpha(B_\ell) + \sum_{i=1}^{N(T)} \delta_{x_i}(B_\ell)}{\alpha(R) + N(T)} \right) \mid N_T, X_T \right\} \\ &= E\{F_T(B_j)F_T(B_\ell) \mid N_T, X_T\} - \frac{\alpha(B_j) + \sum_{i=1}^{N(T)} \delta_{x_i}(B_j)}{\alpha(R) + N(T)} E\{F_T(B_\ell) \mid N_T, X_T\} \\ &\quad - \frac{\alpha(B_\ell) + \sum_{i=1}^{N(T)} \delta_{x_i}(B_\ell)}{\alpha(R) + N(T)} E\{F_T(B_j) \mid N_T, X_T\} \\ &\quad + \frac{\alpha(B_\ell) + \sum_{i=1}^{N(T)} \delta_{x_i}(B_\ell)}{\alpha(R) + N(T)} \cdot \frac{\alpha(B_j) + \sum_{i=1}^{N(T)} \delta_{x_i}(B_j)}{\alpha(R) + N(T)} \\ &= \frac{\left( \alpha(B_j) + \sum_{i=1}^{N(T)} \delta_{x_i}(B_j) \right) \left( \alpha(B_\ell) + \sum_{i=1}^{N(T)} \delta_{x_i}(B_\ell) \right)}{(\alpha(R) + N(T))(\alpha(R) + N(T) + 1)} \\ &\quad - \frac{\alpha(B_j) + \sum_{i=1}^{N(T)} \delta_{x_i}(B_j)}{\alpha(R) + N(T)} \cdot \frac{\alpha(B_\ell) + \sum_{i=1}^{N(T)} \delta_{x_i}(B_\ell)}{\alpha(R) + N(T)} \rightarrow 0, \\ &\quad j \neq \ell, j, \ell = 1, \dots, K. \end{aligned}$$

The case for  $j = \ell$  can be proved similarly. ||



In the following, we shall prove that centered and properly rescaled, the posterior joint distribution of the parameters  $(\lambda, F)$  converges weakly to a joint Gaussian distribution, a.s.  $\left[ P_{\lambda_0, F_0} \right]$ . We let  $F$  be distributions on  $[0,1]$ . Define

$$X_T(t) = \sqrt{N(T)} \left\{ F_T(t) - \frac{\alpha(t) + \sum_{i=1}^{N(T)} \delta_{x_i}(t)}{\alpha(1) + N(T)} \right\}; t \in [0,1]$$

and

$$Y_T = \sqrt{T} \left\{ \lambda_T - \frac{Y + N(T)}{\theta + T} \right\}.$$

Let  $W_0$ ,  $t \in [0,1]$  be the Brownian bridge subject to a change of time by  $F_0$ , i.e.,

$$EW_0(t) = 0, t \in [0,1]$$

$$E\{W_0(t) - W_0(s)\}^2 = \{F_0(t) - F_0(s)\}\{1 - F_0(t) + F_0(s)\}, s \leq t.$$

Let  $Y$  be a centered normal random variable with variance  $\lambda_0$ . Then

Theorem 3.1:

If  $F_0$  is continuous, a.s.  $\left[ P_{\lambda_0, F_0} \right]$

$$(Y_T, X_T(\cdot)) \xrightarrow{L} (Y, W_0(\cdot)) \text{ when } T \rightarrow \infty$$

where  $Y$  and  $W_0(\cdot)$  are independent.

Proof:

Because  $Y_T$  and  $X_T(\cdot)$  are independent, we only need to show  
 (1)  $Y_T \xrightarrow{L} Y$  and (2)  $X_T(\cdot) \xrightarrow{L} W_0(\cdot)$  a.s.  $\left[ P_{\lambda_0, F_0} \right]$ . To show (1),  
 let  $\Omega_1 = \left\{ N : \frac{N(T)}{T} \rightarrow \lambda_0 \right\}$ . This set is of  $P_{\lambda_0, F_0}$  probability 1; see  
 previous arguments. Our assertion will follow if we show that deleting  
 a  $P_{\lambda_0, F_0}$  - null set and for all  $N \in \Omega_1$ , and all  $\{T_n\}$ ,  $T_n \rightarrow \infty$ ,  
 we have  $Y_{T_n} \xrightarrow{L} N(0, \lambda_0)$  as  $n \rightarrow \infty$ . See Billingsley (1968, p. 16).  
 Thus we pick any  $N \in \Omega_1$ , hold  $N$  fixed and then let  $T_n$  be any sequence  
 of positive reals that goes to infinity. We shall show that  $Y_{T_n} \xrightarrow{L} N(0, \lambda_0)$ .

Note that  $Y_{T_n} \stackrel{L}{=} \sum_{i=0}^{N(T_n)} (X_i - EX_i)$  where  $X_j$ ;  $j = 0, 1, \dots, N(T_n)$   
 are independent gamma random variables with distributions  $U_{\gamma_i, \theta + T_n}$   
 respectively with  $\gamma_0 = \gamma$ ,  $\gamma_i = 1$ ;  $1 \leq i \leq N(T_n)$ . Thus, the central  
 limit theorem for triangular array applies and (1) follows.

We show (2) similarly. Let  $\Omega_2 = \left\{ \frac{N(T)}{T} \rightarrow \lambda_0, \frac{\sum_{i=1}^{N(T)} \delta_{X_i}(\cdot)}{N(T)} \xrightarrow{L} F_0(\cdot) \right\}$ .

This set is of  $P_{\lambda_0, F_0}$  probability 1, see Proposition 3.1. Deleting a  
 $P_{\lambda_0, F_0}$  - null set, fix  $(N, X_1, \dots, X_n, \dots) \in \Omega_2$  and let  $T_n$  be a sequence  
 of positive reals that goes to infinity. We need to show  $X_{T_n} \xrightarrow{L} W_0(\cdot)$ ,  
 see Billingsley (1968, p. 16). But then this is proved in Example 1, Part II  
 of Lo (1978). Thus a.s.  $\left[ P_{\lambda_0, F_0} \right]$ ,  $(X_T, X_T(\cdot)) \xrightarrow{L} (Y, W_0(\cdot))$ , where  
 $Y$  and  $W_0(\cdot)$  are independent. ||

The following corollary provides an asymptotic Bayesian simultaneous  
 confidence band for the continuous true distribution functions.

Corollary 3.1:

Under the assumptions of Proposition 3.1, we have a.s.  $\left[ P_{\lambda_0, F_0} \right]$

$$\lim_{T \rightarrow \infty} P \left\{ \sup_{0 \leq t \leq 1} |X_T(t)| > \lambda \right\} = 2 \sum_{j=1}^{\infty} (-1)^{j+1} e^{-2j^2 \lambda^2}, \quad \lambda > 0$$

$$\lim_{T \rightarrow \infty} P \left\{ \sup_{0 \leq t \leq 1} X_T(t) > \lambda \right\} = e^{-2\lambda^2}, \quad \lambda > 0.$$

Proof:

These are well known consequences of Theorem 3.1. For detail arguments, see Billingsley (1968, p. 142).||

#### 4. APPLICATIONS

Let  $f$  be a real valued integrable (or positive) function of  $(\lambda, F)$ , then the Bayes rule given the observations with respect to a quadratic type loss function is  $E\{f(\lambda, F) \mid N_T, X_T\}$ . For different standard functions of  $F$ , the Bayes rules have been computed in Ferguson (1973).

Our concern here is the survival probability  $\bar{H}(t)$ ,  $\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k \cdot \frac{e^{-\lambda t} (\lambda t)^k}{k!}$

where  $\bar{P}_k$  is the probability that the device survives  $k$  shocks for the period  $[0, t]$ . The  $\bar{P}_k$  is a deterministic function of  $k$  and the capacity or threshold of the device. Thus  $\bar{H}(t)$  is the probability that the device survives the period  $[0, t]$ . We consider two cases.

##### Case (1):

In the cumulative damage model, the  $k^{\text{th}}$  shock is survived by the device if  $X_1 + \dots + X_k$  does not exceed the capacity or threshold  $y$  of the device. Note that in this case  $\bar{P}_k = P\{X_1 + \dots + X_k \leq y \mid N(t) = k\}$ . This model has been considered by Cox (1962) and Barlow and Proschan (1975) among others.

##### Case (2a):

$\bar{P}_k = \prod_{i=1}^k P\{X_i \leq y_i\} = \prod_{i=1}^k F(y_i)$ . This represents the case where there is a threshold which changes after each shock occurs and  $y_1, \dots, y_k$  are the successive threshold levels. This model is discussed by Esary et al., (1973).

Case (2b):

$\bar{P}_k = \{F(y)\}^k$ . This is a special case of (2a) with  $y_1 = y_2 = \dots = y_k = y$ , a fixed threshold level.

To compute our Bayes rule for the survival probability, let the loss function be

$$L(\bar{H}, \hat{H}) = \int_{R^+} (\bar{H}(t) - \hat{H}(t))^2 W(dt),$$

where  $W$  is a totally finite measure on  $[0, \infty)$ . Thus, the Bayes rule with respect to this loss function will be, for each  $t$ ,

$$\hat{H}(t) = E\{\bar{H}(t) \mid N_T, X_T\}.$$

Note that  $E\{\bar{H}(t) \mid N_T, X_T\}$  can be calculated using the following properties for the posterior distribution of the parameters.

Property (1a):

$$E\{e^{-\lambda t} \lambda^k \mid N_T, X_T\} = \frac{\Gamma(\gamma + N(T) + k)}{\Gamma(\gamma + N(T))} \left( \frac{1}{\theta + T + t} \right)^k \left( \frac{\theta + T}{\theta + T + t} \right)^{\gamma + N(T)}.$$

Proof:

This is a consequence of Theorem 2.1.

Property (1b):

$$E\{F^{(k)}(y) \mid N_T, X_T\} = \int_{R^+} \dots \int_{R^+} \frac{\left[ \alpha + \sum_{i=1}^{N(T)} \delta_{x_i} + \sum_{j=1}^k \delta_{u_j} \right] \left( y - \sum_{j=1}^k u_j \right)}{\alpha(R^+) + N(T) + k} \prod_{j=1}^k \frac{\left[ \alpha + \sum_{i=1}^{N(T)} \delta_{x_i} + \sum_{l=1}^{j-1} \delta_{u_l} \right] (du_j)}{\alpha(R^+) + N(T) + j - 1}.$$

Proof:

This follows from Theorem 2.1 and repeated applications of Lemma 1 in Lo (1978b).

Property (1c)

$$E \left\{ \prod_{i=1}^k F(y_i) \mid N_T, X_T \right\} = \int_0^{y_1} \cdots \int_0^{y_k} \prod_{j=1}^k \frac{\alpha + \sum_{i=1}^{N(T)} \delta_{x_i} + \sum_{\ell=1}^{j-1} \delta_{u_\ell} (du_j)}{\alpha(R^+) + N(T) + j - 1}.$$

Proof:

This is from Theorem 2.1 and an application of Lemma 1 in Lo (1978b).

Property (1d):

$$E \left\{ F^k(y) \mid N_T, X_T \right\} = \frac{\Gamma(\alpha(R^+) + N(T)) \Gamma \left\{ \alpha(y) + \sum_{i=1}^{N(T)} \delta_{x_i}(y) + k \right\}}{\Gamma(\alpha(R^+) + N(T) + k) \Gamma \left\{ \alpha(y) + \sum_{i=1}^{N(T)} \delta_{x_i}(y) \right\}}.$$

Proof:

This is the  $k^{\text{th}}$  moment of beta random variables. Now, it is easy to see that

$$\begin{aligned} & E\{\bar{H}(t) \mid N_T, X_T\} \\ &= \sum_{k=0}^{\infty} E\left\{ \bar{P}_k \cdot e^{-\lambda t} \lambda^k \cdot \frac{t^k}{k!} \mid N_T, X_T \right\} \text{ by the monotone convergence theorem} \\ &= \sum_{k=0}^{\infty} E\left\{ \bar{P}_k \mid N_T, X_T \right\} \cdot E\left\{ e^{-\lambda t} \lambda^k \mid N_T, X_T \right\} \cdot \frac{t^k}{k!}, \end{aligned}$$

since  $\bar{P}_k$  depends on  $F$  only and the fact that under the posterior distribution,  $F$  and  $\lambda$  are independent.

$$= \sum_{k=0}^{\infty} E \left\{ F^{(k)}(y) \mid N_T, X_T \right\} E \left\{ e^{-\lambda t_\lambda k} \mid N_T, X_T \right\} \cdot \frac{t^k}{k!} \text{ in Case (1)}$$

$$= \sum_{j=0}^{\infty} E \left\{ \prod_{i=1}^k F(y_i) \mid N_T, X_T \right\} E \left\{ e^{-\lambda t_\lambda k} \mid N_T, X_T \right\} \cdot \frac{t^k}{k!} \text{ in case (2a)}$$

$$= \sum_{k=0}^{\infty} E \left\{ F^k(y) \mid N_T, X_T \right\} E \left\{ e^{-\lambda t_\lambda k} \mid N_T, X_T \right\} \cdot \frac{t^k}{k!} \text{ in Case (2b),}$$

and the conditional expectations are given by Property (1a), (1b), (1c) and (1d).

#### 4.1 The Case of Random Threshold

Suppose there is no practical way to inspect the device to determine its threshold  $y$  or  $y_1, \dots, y_k, \dots$ . The threshold must be regarded as a random variable. We let  $y \sim G(dy)$  and in case of  $y_1, \dots, y_k$ , we let these be independent and identically distributed as  $G$ . The distribution function  $G$  is a given threshold distribution for the device. Of course, it might be that  $y_i \sim G_i(dy_i)$ ;  $i = 1, \dots, k$  and  $y_i$ 's are independent. But this case can be treated similarly and we omit the details here. It is not difficult to see that Case (1), Case (2a) and Case (2b) become

Case (1'):

$$\bar{P}_k = \int_0^{\infty} F^{(k)}(y) G(dy) .$$

Case (2a'):

$$\bar{P}_k = \int_0^{\infty} \dots \int_0^{\infty} \prod_{i=1}^k F(y_i) \prod_{i=1}^k G(dy_i)$$

and

Case (2b'):

$$\bar{P}_k = \int_0^{\infty} F^k(y) G(dy) .$$

Now it is easy to see, using Fubini's theorem, that in this model,

$$E \left\{ \int_0^{\infty} F^{(k)}(y) G(dy) \mid N_T, X_T \right\} = \int_0^{\infty} E \left\{ F^{(k)}(y) \mid N_T, X_T \right\} G(dy)$$

$$E \left\{ \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^k F(y_i) \prod_{i=1}^k G(dy_i) \mid N_T, X_T \right\}$$

$$= \int_0^{\infty} \cdots \int_0^{\infty} E \left\{ \prod_{i=1}^k F(y_i) \mid N_T, X_T \right\} \prod_{i=1}^k G(dy_i)$$

$$E \left\{ \int_0^{\infty} F^k(y) G(dy) \mid N_T, X_T \right\} = \int_0^{\infty} E \left\{ F^k(y) \mid N_T, X_T \right\} G(dy) ,$$

and the integrands at the right sides are available in Property (1b), (1c) and (1d).



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## APPENDIX

Proposition 1:

$$\begin{aligned}
 & Q_F\{N_T(A_j) = k_j, j = 1, \dots, l; X_i \in C_i, i = 1, \dots, N(T)\} \\
 (A.1) \quad & = \frac{\Gamma(\gamma + N(T))}{\Gamma(\gamma)} \left(\frac{\theta}{\theta + T}\right)^\gamma \left(\frac{1}{\theta + T}\right)^{N(T)} \cdot \prod_{j=1}^l \frac{m^{k_j}(A_j)}{k_j!} \cdot \prod_{i=1}^{N(T)} F(C_i).
 \end{aligned}$$

$$\begin{aligned}
 & Q_\lambda\{N_T(A_j) = k_j, j = 1, \dots, l; X_i \in C_i, i = 1, \dots, N(T)\} \\
 (A.2) \quad & = e^{-\lambda T} (\lambda T)^{N(T)} \prod_{j=1}^l \frac{m^{k_j}(A_j)}{k_j!} \cdot \int_{\prod_{i=1}^{N(T)} C_i} \prod_{i=1}^{N(T)} \frac{\left[\alpha + \sum_{j=1}^{i-1} \delta_{x_j}\right] (dx_i)}{\alpha(R) + i - 1}.
 \end{aligned}$$

$$\begin{aligned}
 & Q\{N_T(A_j) = k_j, j = 1, \dots, l; X_i \in C_i, i = 1, \dots, N(T)\} \\
 (A.3) \quad & = \frac{\Gamma(\gamma + N(T))}{\Gamma(\gamma)} \left(\frac{\theta}{\theta + T}\right)^\gamma \left(\frac{1}{\theta + T}\right)^{N(T)} \prod_{j=1}^l \frac{m^{k_j}(A_j)}{k_j!} \cdot \int_{\prod_{i=1}^{N(T)} C_i} \prod_{i=1}^{N(T)} \frac{\left[\alpha + \sum_{j=1}^{i-1} \delta_{x_j}\right] (dx_i)}{\alpha(R) + i - 1}.
 \end{aligned}$$

Proof:

(A.1) is obtained by integrating (2.1) with respect to  $U_{\gamma, \theta}$ . To show (A.2) and (A.3), we first integrate (2.1) with respect to  $P_\alpha$  and  $U_{\gamma, \theta} \times P_\alpha$  respectively and then an application of Lemma 1 in Lo (1978b) concludes the proof. ||